Permutation, Combination and Probability (1)

1. **(a)** In how many ways can a student answers 8 true-false questions ?
(b) In how many ways may the test be completed if a student is imposed for each incorrect answer, so that the student may leave some questions unanswered?

(a) Number of ways = $2^8 = \mathbf{256}$
(b) Number of ways = $3^8 = \mathbf{6561}$
(There are 3 choices for each question, correct, not correct, unanswered.)

2. In how many ways can a committee of 2 Englishmen, 2 Frenchman, 1 American be chosen from 6 Englishmen, 7 Frenchm 3 American ? In how many ways do a particular Englishman and a particular Frenchman belong to a committee ?

Number of ways to form a committee = $\mathrm{C}(6,2) \times \mathrm{C}(7,2) \times \mathrm{C}(3,1) = \mathbf{945}$
Number of ways a particular Englishman and a particular Frenchman belong to a committee
$= \mathrm{C}(5,1) \times \mathrm{C}(6,1) \times \mathrm{C}(3,1) = \mathbf{90}$

3. A company has 12 construction workers. The manager plans to assign 3 to job site A, 4 to job site B and 5 to job site C. In how many different ways can the manager make this assignment?

12 construction workers and each worker can only be assigned to one of the three sites.

The number of different ways = $\frac{12!}{3!4!5!} = \mathbf{27720}$

4. How many different arrangements of four letters in a row that can be made from the letters of the word (a) "COMBINE" (b) "PROBABILITY"

(a) Number of different arrangements = $\mathrm{P}(7,4) = 7 \times 6 \times 5 \times 4 = \mathbf{840}$
(b) There are 2 'B's and 2 'I's. There are 11 letters. There are 7 which are not B or I.

Ways (no B and no I) = $\mathrm{P}(7,4) = 840$
Ways (1B, no I) = Ways(1I, no B) = $\mathrm{C}(7,3) \times 4! = 840$
Ways (1B, 1I) = $\mathrm{C}(7,2) \times 4! = 504$

Ways (2B, no I) = Ways (2I, no B) = $\mathrm{C}(7,2) \times \frac{4!}{2} = 252$

Ways (2B, one I) = Ways (2I, one B) = $\mathrm{C}(7,1) \times \frac{4!}{2} = 84$

Ways (2B, 2I) = $\frac{4!}{2!2!} = 6$

Total number of ways = $840 + 840 \times 2 + 504 + 252 \times 2 + 84 \times 2 + 6 = \mathbf{3702}$

5. A bag contains 5 green marbles, 4 blue marbles and 6 red marbles. A marble is picked at random. Without replacing the first marble, another marble is taken from the bag. Calculate the probability that

(a) the first marble is green and the second marble red.

(b) two marbles are NOT of the same colour.

(a) $\mathrm{P(G_1)} = \frac{5}{5+4+6} = \frac{5}{15} = \frac{1}{3}$, $\mathrm{P(R_2|G_1)} = \frac{6}{4+4+6} = \frac{6}{14} = \frac{3}{7}$

$\mathrm{P(G_1 \text{ and } R_2) = P(G_1)\ P(R_2|G_1)} = \frac{1}{3} \times \frac{3}{7} = \boldsymbol{\frac{1}{7}}$

(b) P(two marbles are not the same colour)

$= \mathrm{P(G_1)\ P(G_2|G_1) + P(B_1)\ P(B_2|B_1) + P(R_1)\ P(R_2|R_1)}$

$= \frac{5}{5+4+6} \times \frac{4}{4+4+6} + \frac{4}{5+4+6} \times \frac{3}{5+3+6} + \frac{6}{5+4+6} \times \frac{5}{5+4+5} = \frac{31}{105}$

P(two marbles are not the same colour) $= 1 -$ P(two marbles are not the same colour)

$= 1 - \frac{31}{105} = \boldsymbol{\frac{74}{105}}$

6. In arranging a 10-day examination time-table involving 10 subjects and one subject per day, a teacher plans to have Mathematics, Physics and Chemistry all separated by at least one day. How many ways are possible?

U = universal set of all possible arrangement

M = Mathematics, P = Physics, C = Chemistry

MP = M and P joined in consecutive days.

MPC = P, M and C joined in consecutive days.

$|\mathrm{U}| = 10!$

$|\mathrm{MP}| = \frac{9!}{2!}$, $|\mathrm{PC}| = \frac{9!}{2!}$, $|\mathrm{CM}| = \frac{9!}{2!}$ (take MP as one subject for 2 consecutive days)

$|\mathrm{MPC}| = \frac{8!}{3!}$ (take MPC as one subject for 3 consecutive days)

If P, M, C are separated by one day, possible ways

$= |\mathrm{U}| - (|\mathrm{MP}| - |\mathrm{MPC}|) - (|\mathrm{PC}| - |\mathrm{MPC}|) - (|\mathrm{CM}| - |\mathrm{MPC}|) - |\mathrm{MPC}|$

$= |\mathrm{U}| - |\mathrm{MP}| - |\mathrm{PC}| - |\mathrm{CM}| + 2|\mathrm{MPC}| = 10! - \frac{9!}{2!} - \frac{9!}{2!} - \frac{9!}{2!} + \frac{8!}{3!} = \mathbf{3091200}$

7. 0000, 0001, 0002, …, 9999 are ten thousand 4-digits numbers. The numbers are classified into the following groups,

(a) All 4 digits are the same.

(b) Three digits are the same and the remaining digit is different.

(c) Two pairs of the same digits
(d) One pair of the same digits and the other two digits are different.
(e) All digits are different.

Calculate the number of numbers in each group.

(a) **10**

(b) If the three same digits is 0, the other digits are 1, 2, …, 9 can be placed in unit, ten, hundred , thousand place.

The number of numbers = 9×4

This is the same if the three digits is 1, 2, …, 9.

Hence, total number of numbers = $9 \times 4 \times 10 = \mathbf{360}$

(c) Choose any two digits from 10 digits, combination = $C(10,2) = 45$

The number of ways to place this selected digits in unit, ten, hundred , thousand place

$= \frac{4!}{2!2!} = 6$

Hence, total number of numbers = $45 \times 6 = \mathbf{270}$

(d) There are 10 ways to choose the paired digits and there are $C(9,2) = 36$ ways to choose the remaining digits.

The number of ways to place this selected digits in unit, ten, hundred , thousand place

$= \frac{4!}{2!} = 12$

Hence, total number of numbers = $10 \times 36 \times 12 = \mathbf{4320}$

(e) Choose any 4 digits from 10 digits, combination = $C(10,4) = 210$

The number of ways to place this selected digits in unit, ten, hundred , thousand place

$= 4! = 24$

Hence, total number of numbers = $210 \times 24 = \mathbf{5040}$

Checking :

(a) + (b) + (c) + (d) + (e)

$$= C(10,1)\frac{4!}{4!} + C(10,1)C(9,1)\frac{4!}{3!1!} + C(10,2)\frac{4!}{2!2!} + C(10,1)C(9,2)\frac{4!}{2!1!1!} + C(10,4)\frac{4!}{1!1!1!1!}$$

$$= \mathbf{10 + 360 + 270 + 4320 + 5040 = 10000}$$

8. A production process uses two machines in its daily production. A random sampling produced are inspected and the following contingency table is obtained

	Defective	Non-defective
Machine X	15	285
Machine Y	6	194

If an item is selected randomly, what is the probability that the item is

(a) defective

(b) produced by machine X and defective,

(c) produced by machine X or non-defective,

(d) defective given that it is produced by machine X

$P(D|X) = \frac{15}{15+285} = \frac{1}{20}, \quad P(\overline{D}|X) = \frac{285}{15+285} = \frac{19}{20},$

$P(D|Y) = \frac{6}{6+194} = \frac{3}{100}, \quad ,P(\overline{D}|Y) = \frac{194}{6+194} = \frac{97}{100}$

(a) $P(\overline{D}) = P(X)P(\overline{D}|X) + P(Y)P(\overline{D}|Y) = \frac{1}{2} \times \frac{19}{20} + \frac{1}{2} \times \frac{97}{100} = \frac{24}{25}$

(b) $P(X \text{ and } \overline{D}) = P(X)P(\overline{D}|X) = \frac{1}{2} \times \frac{19}{20} = \frac{19}{40}$

(c) **Method 1**

$P(Y \text{ and } \overline{D}) = P(Y)P(\overline{D}|Y) = \frac{1}{2} \times \frac{97}{100} = \frac{97}{200}$

$P(X \text{ and } D) = P(X)P(D|X) = \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$

$P(Y \text{ and } D) = P(Y)P(D|Y) = \frac{1}{2} \times \frac{3}{100} = \frac{3}{200}$

$P(X \text{ or } \overline{D}) = P(X \text{ and } \overline{D}) + P(X \text{ and } D) + P(Y \text{ and } \overline{D}) = \frac{19}{40} + \frac{1}{40} + \frac{97}{200} = \frac{197}{200} = \mathbf{0.985}$

Method 2

$P(X \text{ and } D) = P(X)P(D|X) = \frac{1}{2} \times \frac{1}{20} = \frac{1}{40}$

$P(X \text{ or } \overline{D}) = P(\overline{D}) + P(X \text{ and } D) = \frac{24}{25} + \frac{1}{40} = \frac{197}{200} = \mathbf{0.985}$

Method 3

$P(X \text{ or } \overline{D}) = P(X) + P(\overline{D}) - P(X \text{ and } \overline{D}) = P(X) + P(\overline{D}) - P(X)P(\overline{D}|X)$

$= \frac{1}{2} + \frac{24}{25} - \frac{1}{2} \times \frac{19}{20} = \frac{197}{200} = \mathbf{0.985}$

(d) $P(D|X) = \frac{1}{20}$

9. A navigation signal is made of flags arranged in a row. If there are 4 red flags, 2 blue flags and 2 green flags, find the number of different signals possible if

(a) we can use all the flags

(b) at least 7 flags must be used for the signal.

(a) Number of different signals possible $= \frac{(4+2+2)!}{4!2!2!} = \mathbf{420}$

(b) Number of different signals possible if 7 flags are used

$= \text{Arrange}(4R, 2B, 1G) + \text{Arrange}(4R, 1B, 2G) + \text{Arrange}(4R, 2B, 1G) + \text{Arrange}(3R, 2B, 2G)$

$= \frac{(4+2+1)!}{4!2!1!} + \frac{(4+1+2)!}{4!1!2!} + \frac{(3+2+2)!}{3!2!2!} = 420$

The number of different signals possible if at least 7 flags must be used for the signal

$= 420 + 424 = \mathbf{840}$

10. A group of students sit for both the Mathematics and Physics papers in school examination. Their results are summarized as follow:

75% pass Mathematics

70% passes Physics

40% fail in at least one of the subjects.

A student is selected randomly from the group.

(a) Find the probability that the student passes only one of the two subjects.

(b) Among those who pass Mathematics, find the probability that they also pass Physics.

(a) $P(M) = 0.75, P(P) = 0.7, P(M' \cup P') = 0.4$

$P(M' \cup P') = P\big((M \cap P)'\big) \Longrightarrow P(M \cap P) = 1 - 0.4 = 0.6$

$P(M \,\Delta\, P) = P(M) + P(P) - 2P(M \cap P) = 0.75 + 0.7 - 2(0.6) = \mathbf{0.25}$

(b) $P(P|M) = \frac{P(P \cap M)}{P(M)} = \frac{0.6}{0.75} = \mathbf{0.8}$

11. In how many ways can a committee of 3 women and 4 men are chosen from 8 women and 7 men? What is the number of ways if Miss X refuses to serve if Mr. Y is a member?

Number of ways to form a committee $= C(8,3) \times C(7,4) = 56 \times 35 = \mathbf{1960}$

Number of ways if Miss X refuses to serve if Mr. Y is a member

$= \text{Ways}(X \cap Y') + \text{Ways}(X' \cap Y) + \text{Ways}(X' \cap Y')$

$= C(7,2) \times C(6,4) + C(7,3) \times C(6,3) + C(7,3) \times C(6,4) = \mathbf{1540}$

12. Find the number of permutations that can be formed from the letters of the word POPULAR. How many of these permutations:

(a) begin and end with P?

(b) have the two P's separated?

(c) have the vowels together?

(a) Since the P's are fixed, the other 5 letters can be permutated.

Number of permutations $= 5! = \mathbf{120}$

(b) If the two P's must be placed together, let this two P's are joined as one letter (PP), so the number of permutations $= (7-1)! = 720$

Total permutations with the two P's joined or not joined together $= \frac{7!}{2!} = 2520$

Number of permutations with the two P's separated $= 2520 - 720 = \mathbf{1800}$

(c) There are 3 vowels O,U,A , let them joined together as one letter (OUA), so there are 5 letters

{(OUA), P,P, L, R} , number of permutations $= \frac{5!}{2!} = 60$

However the vowels O,U,A can be permutated and the number of permutations $= 3! = 6$

So the number of permutations $= 60 \times 6 = \mathbf{360}$

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